Spectra of Random Operators with absolutely continuous Integrated Density of States

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Dedicated to Prof. Fritz Gesztesy on the occasion of his 60th Birthday

Abstract

The structure of the spectrum of random operators is studied. It is shown that if the density of states measure of some subsets of the spectrum is zero, then these subsets are empty. In particular follows that absolute continuity of the IDS implies singular spectra of ergodic operators is either empty or of positive measure. Our results apply to Anderson and alloy type models, perturbed Landau Hamiltonians, almost periodic potentials and models which are not ergodic.

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1 Introduction

Here we study some aspects about the structure of the almost sure spectrum of random operators. This was inspired by Barbieri et al. [2] where it was shown for some ergodic operators of Anderson type, that their almost sure singular continuous spectrum Σ_{sc} satisfies either $|\Sigma_{sc} \cap J| > 0$ or $\Sigma_{sc} \cap J = \emptyset$, where $|\cdot|$ denotes the Lebesgue measure and $J \subset \mathbb{R}$ is any interval. This is not true for every ergodic operator. The Fibonacci model for example, see [19], has nonempty almost sure singular continuous spectrum with zero Lebesgue measure. So, for which models and for which kinds of spectra the result of Barbieri et al. holds?.In this work we show that an answer can be given through the integrated density of states, IDS for short, of the corresponding operators.

In [2] nothing is mentioned about IDS and the method used rely on Howland's theory on relative finite perturbations. If the IDS were absolutely continuous the result in [2] will follow immediately from Corollary 1 below. The so called Wegner estimates imply regularity of the IDS and often its absolute continuity. There has been a lot of effort spent on proving these estimates, particularly because they provide a key step in some methods for proving localization [18]. I have not found in the literature a proof of the absolute continuity of the IDS with the exact conditions on the model given in [2], but there are results on absolute continuity of the IDS for closely related models. In [3] Corollary 4.6, the authors prove absolute continuity for the IDS with conditions very similar to the ones in [2]. In this case the nonexistence of almost sure spectra of zero measure, in particular singular continuous spectrum, can be proved with techniques that depend on the behavior of the IDS as we show in what follows.

Knowledge of the IDS can give us then information about the measure of almost sure spectra. Our main theorem 4 says that if the density of states measure of a particular spectrum in an interval is zero, then this spectrum is empty inside that interval. This holds for $\mathbb{P}$-positive spectra (see definition 1 below), in particular for almost sure spectra, which could be singular continuous, pure point etc.. Using the absolute continuity of the IDS that follows from Wegner estimates proved by several authors for different models, we shall then see for a variety of situations, including Anderson and alloy

type models, perturbed Landau Hamiltonians, almost periodic potentials and even models which are not ergodic, that any almost sure spectra has positive measure, if it is not empty. In section 2 we present basic definitions about random operators which are required and mention some important theorems about the existence of almost sure spectra for ergodic operators. Here the integrated density of states is introduced. In section 3, we present the main results. These will allows us to use the absolute continuity of the IDS to prove the mentioned statement about the measure of the almost sure spectra. We finish by giving some explicit examples where our results can be applied.

2 Preliminaries

Let $(\Omega, \mathcal{M}, \mathbb{P})$ be a complete probability space . By $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ we shall denote the sets of integer, real and complex numbers respectively. The scalar product in a Hilbert space will be denoted by $\langle , \rangle$.

We need following definitions. See [10].

Definition 1. A family of bounded operators $\{H_\omega\}_{\omega\in\Omega}$ defined on Hilbert space $\mathcal{H}$ is *weakly measurable*, if $\Omega \ni \omega \longrightarrow \langle x, H_\omega y\rangle \in \mathbb{C}$ is measurable for every $x, y \in \mathcal{H}$. A family of selfadjoint operators $\{H_\omega\}_{\omega\in\Omega}$ is *measurable* , if $\omega \longrightarrow (H_\omega - z)^{-1}$ is weakly measurable for all $z \in \mathbb{C}\backslash\mathbb{R}$.

Definition 2. A family of measurable transformations $T_i : \Omega \to \Omega$, $i \in \mathbb{Z}^d$ is called *measure preserving* if $\mathbb{P}(T_i^{-1}A) = \mathbb{P}(A)$ for every $A \in \mathcal{M}$ and *ergodic* if $T_i^{-1}A = A$ for all $i \in \mathbb{Z}^d$ implies $\mathbb{P}(A) = 0$ or 1.

Definition 3. Let $\{T_i\}_{i\in\mathbb{Z}^d}$ be measure preserving and ergodic. A measurable family of selfadjoint operators $\{H_\omega\}_{\omega\in\Omega}$ on a separable Hilbert space $\mathcal{H}$ is called $\mathbb{Z}^d$ *ergodic* if there exist a family $\{U_i\}_{i\in\mathbb{Z}^d}$ of unitary operators in $\mathcal{H}$ such that $H_{T_i\omega} = U_i H_\omega U_i^*$. We call the family $\{H_\omega\}_{\omega\in\Omega}$ $\mathbb{Z}^d$ *stationary*, if we just require $\{T_i\}_{i\in\mathbb{Z}^d}$ to be measure preserving, (may be ergodic too).

The following theorem was proven by L. Pastur [16]. $\sigma(H_\omega)$ denotes the spectrum of H_ω

Theorem 1. . *Let $\{H_\omega\}_{\omega\in\Omega}$ be an ergodic family.*

There exists $\Sigma \subset \mathbb{R}$ such that $\sigma(H_\omega) = \Sigma$ for $\mathbb{P}$ almost all $\omega \in \Omega$, that is for all $\omega \in \Omega_1$ with $\mathbb{P}(\Omega_1) = 1$.(Σ is ω independent).

Analogous results hold for other parts of the spectrum. See [12], [13]

Theorem 2. *Let $\{H_\omega\}_{\omega\in\Omega}$ be an ergodic family. There exist ω independent sets $\Sigma_{ac}, \Sigma_{sc}, \Sigma_{pp}$ such that*
$\Sigma_{ac} = \sigma_{ac}(H_\omega)$ $\Sigma_{sc} = \sigma_{sc}(H_\omega)$ $\Sigma_{pp} = \sigma_{pp}(H_\omega)$ *for $\mathbb{P}$ almost all ω .*

The sets $\sigma_{ac}(H_\omega), \sigma_{sc}(H_\omega), \sigma_{pp}(H_\omega)$, denote the absolutely continuous, singular continuous and pure point spectra of H_ω as defined in [20] p. 106, [14] or [9] section 7.2 .The pure point spectrum $\sigma_{pp}(H_\omega)$is the closure of the set of the eigenvalues of H_ω.

Finer decompositions of the spectra are possible. For ergodic operators in $l^2(\mathbb{Z}^d)$ the following result was proven in [14]) (theorem 8.1),

Theorem 3. *For $\alpha \in [0,1]$ there exist subsets of $\mathbb{R}$: $\sigma_{\alpha ds}, \sigma_{ed\alpha/\alpha s}, \sigma_{\alpha ac}, \sigma_{ed\alpha/s\alpha c}$ and $\sigma_{s\alpha dc}$ such that for $\mathbb{P}$ almost all ω they are respectively the α-dimension singular , α-singular of exact dimension α , absolutely continuous with respect to h^α , strongly α-continuous of exact dimension α and strongly α-dimension continuous spectra of H_ω*

For the definition of all this kinds of different spectra , see [14].

If H is an operator with domain $D(H) \subset \mathcal{H}$ in Hilbert space $\mathcal{H}$ and $M \subset \mathcal{H}$ is a subspace of $\mathcal{H}$, the *restriction of H to M* denoted by $H|_M$ is the operator with domain $D(H|_M) = M \cap D(H)$ and such that for $f \in D(H|_M)$ one has $H|_M f = Hf$.

Let P_M be the orthogonal projection on the closed subsapce $M \subset \mathcal{H}$ of the Hilbert space $\mathcal{H}$. M is said to *reduce* the symmetric operator H or to be a *reducing subspace for H* if $u \in D(H)$ implies $P_M u \in D(H)$ and $HP_M u \in M$. See for example [8] p. 278.

Definition 4. Assume $M_\omega \subseteq \mathcal{H}$ reduces the operator H_ω. A set $\Sigma \subset \mathbb{R}$ is called an *almost sure spectrum* for $\{H_\omega\}_{\omega\in\Omega}$ if there exist a set $\Omega_\Sigma \subset \Omega$ with $\mathbb{P}(\Omega_\Sigma) = 1$ such that $\Sigma = \sigma(H_\omega|_{M_\omega})$ for all $\omega \in \Omega_\Sigma$, that is for $\mathbb{P}$ almost all ω. (Σ is ω independent). If $\mathbb{P}(\Omega_\Sigma) > 0$ then we call Σ a $\mathbb{P}$-*positive spectrum* for $\{H_\omega\}_{\omega\in\Omega}$.

Remark 1. From Definition 1 it follows that for $\mathbb{P}$ almost all $\omega \in \Omega$

$$\sigma(H_\omega), \sigma_{sc}(H_\omega), \sigma_{pp}(H_\omega), \sigma_{ac}(H_\omega), \sigma_{\alpha ds}(H_\omega),$$
$$\sigma_{ed\alpha/\alpha s}(H_\omega), \sigma_{\alpha ac}(H_\omega), \sigma_{ed\alpha/s\alpha c}(H_\omega), \sigma_{s\alpha}(H_\omega)$$

are almost sure spectra for $\{H_\omega\}_{\omega\in\Omega}$ if this family is ergodic.

There can be almost sure spectra for operators which are not ergodic. See for example [15] Corollary 1.1.3.

Now let us consider an $\mathbb{Z}^d$ stationary family of selfadjoint operators H_ω acting on $L_2(\mathbb{R}^d)$ or $l_2(\mathbb{Z}^d)$. Let $\Lambda = [-1/2, 1/2]^d \cap \mathbb{R}^d$ and denote by χ_Λ the characteristic function of the set Λ, that is $\chi_\Lambda(x) = 1$ if $x \in \Lambda$ and $\chi_\Lambda(x) = 0$ if $x \notin \Lambda$.

In case H_ω acts in $l_2(\mathbb{Z}^d)$ we talk of *the discrete model* and define

$$\nu(A) := \mathbb{E}(\langle \delta_0, E_{H_\omega}(A)\delta_0 \rangle) \tag{1}$$

and in case H_ω acts in $L_2(\mathbb{R}^d)$ we talk of the *continuous model* and define

$$\nu(A) := \mathbb{E}(tr \chi_\Lambda E_{H_\omega}(A)\chi_\Lambda) \tag{2}$$

for any Borel set A. $\mathbb{E}$ denotes the *mathematical expectation*, that is $\mathbb{E}(\dots) = \int_\Omega \dots d\mathbb{P}$. The symbol δ_0 denotes the function such that $\delta_0(0) = 1$ and $\delta_0(n) = 0$ for any $n \in \mathbb{Z}^d, n \neq 0$. In general we define δ_i as $\delta_i(n) = 1$ if $i = n$ and $\delta_i(n) = 0$ otherwise. $E_{H_\omega}(A)$ is the *spectral projection measure* associated to the selfadoint operator H_ω, see [8] p.355, that is $E_{H_\omega}(A) = \chi_A(H_\omega)$ where χ_A is the characteristic function of the Borel set $A \subset \mathbb{R}$. By tr we denote the trace , which is uniquely defined (may be $+\infty$) for any bounded positive operator B as $tr B = \sum_{n=1}^{\infty} \langle \varphi_n, B\varphi_n \rangle$ where $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal basis. In equation (2) χ_Λ is understood as a multiplication operator.

The measure $\nu(A)$ defined above is called *the density of states measure.* The distribution function N of ν defined by

$$N(E) = \nu((-\infty, E]) \tag{3}$$

is known as *the integrated density of states* IDS. See [9]. We shall use the short hand notation IDS for the density of states measure or for the integrated density of states. For more information on this object the interested reader can see [21] and [11].

Observe that the function N is absolutely continuous if and only if the measure ν is absolutely continuous.

3 Main results

Let $\{H_\omega\}_{\omega \in \Omega}$ be a $\mathbb{Z}^d$ stationary family of selfadjoint operators acting on $L_2(\mathbb{R}^d)$ or $l_2(\mathbb{Z}^d)$ where the corresponding unitary operators are given by $(U_i\varphi)(x) = \varphi(x - i), i \in \mathbb{Z}^d$ for $\varphi \in L_2(\mathbb{R}^d)$ or $l_2(\mathbb{Z}^d)$. Let $I \subset \mathbb{R}$ be a closed interval which may be unbounded and denote its interior by I°. Our main theorem is

Theorem 4. *If Σ is a $\mathbb{P}$-positive spectrum for $\{H_\omega\}_{\omega\in\Omega}$, then $\nu(\Sigma\cap I)=0$ implies $\Sigma\cap I^\circ=\emptyset$, where ν is the density of states measure.*

For the proof we shall need Lemmas 1, 2 and 3 which are stated later in this section.

Proof. A) *Discrete model.*

First step.

From Lemma 1 we know that

$$\nu(\Sigma)=\mathbb{E}(\langle\delta_i,E_{H_\omega}(\Sigma\cap I)\delta_i\rangle)$$

for all $i\in\mathbb{Z}^d$. If we assume that $\nu(\Sigma\cap I)=0$, then for each fixed $i\in\mathbb{Z}^d$ there exists a set $B_i\subset\Omega$ with $\mathbb{P}(B_i)=0$ such that for all $\omega\in\Omega\backslash B_i$ we have $\langle\delta_i,E_{H_\omega}(\Sigma\cap I)\delta_i\rangle=0$. Consider the set $\Omega':=\Omega\backslash(\cup_{i\in\mathbb{Z}^d}B_i)$, then $\mathbb{P}(\Omega')=1$ and for all $\omega\in\Omega'$,

$$\langle\delta_i,E_{H_\omega}(\Sigma\cap I)\delta_i\rangle=\parallel E_{H_\omega}(\Sigma\cap I)\delta_i\parallel^2=0 \tag{4}$$

Using expression (4) we can see that for any $f\in l^2(\mathbb{Z}^d)$

$$\parallel E_{H_\omega}(\Sigma\cap I)f\parallel^2=\langle f,E_{H_\omega}(\Sigma\cap I)f\rangle=0 \tag{5}$$

for all $\omega\in\Omega'$. For this it is enough to write f in the basis $\{\delta_i\}_{i\in\mathbb{Z}^d}$, substitute in (5) and recall that $E_{H_\omega}(\Sigma\cap I)\delta_i$ for every i is the zero vector by (4). Then

$$\parallel E_{H_\omega}(\Sigma\cap I)f\parallel^2=\parallel E_{H_\omega}(\Sigma\cap I)(\sum_i c_i\delta_i)\parallel^2=\parallel\sum_i c_iE_{H_\omega}(\Sigma\cap I)\delta_i\parallel^2=0$$

for all $\omega\in\Omega'$.

Now, since Σ is a $\mathbb{P}$- positive spectrum for $\{H_\omega\}_{\omega\in\Omega}$, (see Definition 1), there exists a set Ω_Σ with $\mathbb{P}(\Omega_\Sigma)>0$ such that $\Sigma=\sigma(H_\omega|_{M_\omega})$ for all $\omega\in\Omega_\Sigma$

Set $\tilde\Omega:=\Omega'\cap\Omega_\Sigma$. Then $\mathbb{P}(\tilde\Omega)>0$ and therefore $\tilde\Omega\neq\emptyset$ and from (5) we get

$$\parallel E_{H_\omega}\big(\sigma(H_\omega|_{M_\omega})\cap I\big)f\parallel^2=0 \tag{6}$$

for any $f\in l^2(\mathbb{Z}^d)$ if we require $\omega\in\tilde\Omega$.

Second step.

Fix $\omega_0\in\tilde\Omega$ and use the shorthand notation $S:=H_{\omega_0}|_{M_{\omega_0}}$. By Lemma 3, the set $M:=RangE_S(I)$ is a reducing subspace for S. The subspace $M\subset M_{\omega_0}$ is a reducing subspace for H_{ω_0} too. One way to see this is to observe, using Lemma 2, that the orthogonal projection P_M onto the subspace

M is given by $P_M = E_{H_{\omega_0}}(I)P_{M_{\omega_0}}$ where $P_{M_{\omega_0}}$ is the orthogonal projection onto M_{ω_0} and then notice that $P_M E_{H_{\omega_0}}(t) = E_{H_{\omega_0}}(t)P_M$ for all $t \in \mathbb{R}$. Commutation of the projection with the spectral family is known to be equivalent to reducibility, see Lemma 3 and [24] thm. 7.28. Now

$$\| E_{H_{\omega_0}}\sigma(H_{\omega_0}|_M)f \|^2 = \| E_{H_{\omega_0}}\sigma(S|_M)f \|^2 \leq \| E_{H_{\omega_0}}(\sigma(S)\cap I)f \|^2 = 0 \tag{7}$$

for every $f \in l_2(\mathbb{Z}^d)$. The first equality holds because from the definition of restriction of an operator to a subspace we have $S|_M = \big(H_{\omega_0}|_{M_{\omega_0}}\big)|_M = H_{\omega_0}|_M$. The inequality follows from Lemma 3 and the last equality from (6).

If $f \in M$,

$$E_{H_{\omega_0}}\big(\sigma(H_{\omega_0}|_M)\big)f = E_{H_{\omega_0}|_M}\big(\sigma(H_{\omega_0}|_M)\big)f = f$$

The first equality follows from Lemma 2 and for the second, recall that $E_T(\sigma(T)) = id$, for any selfadjoint operator T, see for example Corollary 3.9 [20]. Using (7) then we conclude that $M = \{0\}$ and therefore

$$\sigma(H_{\omega_0}|_M) = \emptyset$$

. Hence

$$\Sigma \cap I^\circ = \sigma(H_{\omega_0}|_{M_{\omega_0}}) \cap I^\circ = \sigma(S) \cap I^\circ \subset \sigma(S|_M) = \sigma(H_{\omega_0}|_M) = \emptyset$$

. The contention follows from Lemma 3.

B) *Continuous model.*

First step

Assume $\nu(\Sigma \cap I) = 0$. Then

$$\nu(\Sigma \cap I) = \mathbb{E}(tr\chi_\Lambda E_{H_\omega}(\Sigma \cap I)\chi_\Lambda) = \mathbb{E}(tr\chi_{\Lambda_i} E_{H_\omega}(\Sigma \cap I)\chi_{\Lambda_i}) = 0$$

for every $i \in \mathbb{Z}^d$, according to Lemma 1, where $\Lambda_i := [-1/2+i, i+1/2]^d$. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $L_2(\mathbb{R}^d)$ and fix $i \in \mathbb{Z}^d$. Then

$$\begin{aligned}
\mathbb{E}(tr\chi_{\Lambda_i} E_{H_\omega}(\Sigma \cap I)\chi_{\Lambda_i}) &= \mathbb{E}\Bigg(\sum_{n=1}^{\infty}\langle\varphi_n, \chi_{\Lambda_i} E_{H_\omega}(\Sigma \cap I)\chi_{\Lambda_i}\varphi_n\rangle\Bigg) = \\
&= \mathbb{E}(\Bigg(\sum_{n=1}^{\infty} \| E_{H_\omega}(\Sigma \cap I)\chi_{\Lambda_i}\varphi_n \|^2\Bigg) = 0
\end{aligned}$$

and therefore there exists a set $B_i \subset \Omega$ with $\mathbb{P}(B_i) = 0$ such that

$$\sum_{n=1}^{\infty} \| E_{H_\omega}(\Sigma \cap I)\chi_{\Lambda_i}\varphi_n \|^2 = 0 \tag{8}$$

for every $\omega \in \Omega \backslash B_i$. Now take $f \in L_2(\mathbb{R}^d)$ and write f with respect to the basis $\{\varphi_n\}_{n\in\mathbb{N}}$. We get, for any fixed $i \in \mathbb{Z}^d$

$$\| E_{H_\omega}(\Sigma\cap I)\chi_{\Lambda_i} f \| = \| E_{H_\omega}(\Sigma\cap I)\chi_{\Lambda_i}(\sum_{n=1}^{\infty} c_n\varphi_n) \| = \| \sum_{n=1}^{\infty} c_n E_{H_\omega}(\Sigma\cap I)\chi_{\Lambda_i}\varphi_n \| = 0 \tag{9}$$

for all $\omega \in \Omega \backslash B_i$.

The second equality follows from the continuity of the operators $E_{H_\omega}(\Sigma \cap I)$ and χ_{Λ_i} and the last equality because $E_{H_\omega}(\Sigma \cap I)\chi_{\Lambda_i}\varphi_n$ is the zero vector for all n almost surely, which follows from (8). Observe that $f = \sum_{i\in\mathbb{Z}^d} \chi_{\Lambda_i} f$. Then,

$$\langle f, E_{H_\omega}(\Sigma \cap I) f\rangle = \langle f, E_{H_\omega}(\Sigma \cap I) \sum_{i\in\mathbb{Z}^d} \chi_{\Lambda_i} f\rangle = \langle f, \sum_{i\in\mathbb{Z}^d} E_{H_\omega}(\Sigma \cap I)\chi_{\Lambda_i} f\rangle = 0 \tag{10}$$

for every $\omega \in \Omega' := \Omega \backslash (\cup_{i\in\mathbb{Z}^d} B_i)$.

Second step

Follows as in the discrete case A). □

As special case of theorem 4 we have the following

Corollary 1. *Let $\{H_\omega\}_{\omega\in\Omega}$ be as in theorem 4. Assume moreover that this family is ergodic. If the density of states measure $\nu(\cdot)$ is absolutely continuous with respect to a measure $\gamma(\cdot)$ then $\gamma(I \cap \sigma_\star) = 0$ implies $I^\circ \cap \sigma_\star = \emptyset$, where I is any closed interval and $\sigma_\star$ is any almost sure spectrum for H_ω. We could take for example $\star = sc, pp, ac$ or any of the almost sure spectra mentioned in remark 1.*

The following lemmas are more or less standard facts.

Lemma 1. *Let $\{H_\omega\}_{\omega\in\Omega}$ be as theorem 4. Then, for every $i \in \mathbb{Z}^d$*

a) $\nu(\cdot) = \mathbb{E}(tr\chi_{\Lambda_i} E_{H_\omega}(\cdot)\chi_{\Lambda_i})$ in the continuous case where $\Lambda_i := [-1/2 + i, i + 1/2]^d$. We shall write Λ for Λ_0.

b) $\nu(\cdot) = \mathbb{E}(\langle\delta_i, E_{H_\omega}(\cdot)\delta_i\rangle)$ in the discrete case.

Proof. First recall two facts: If h is a measurable function and T_i is measure preserving, then

$$\mathbb{E}(h(T_i\omega)) = \int_\Omega h(T_i\omega)d\mathbb{P}(\omega) = \int_\Omega h(\omega)d\mathbb{P}(T_i^{-1}\omega) = \int_\Omega h(\omega)d\mathbb{P}(\omega) = \mathbb{E}(h(\omega)) \tag{11}$$

. See for example [17] p. 13.

The second is :

If S is a selfadjoint operator and U unitary operator, then for any bounded measurable function f we have

$$f(USU^*) = Uf(S)U^* \tag{12}$$

. See Lemma 4.5 of [9].

Case a). Continuous model.

Let $h(\omega) := tr\chi_\Lambda E_{H_\omega}(A)\chi_\Lambda$. Then from (11), for every $i \in \mathbb{Z}^d$,

$$\mathbb{E}(tr\chi_\Lambda E_{H_\omega}(\cdot)\chi_\Lambda) = \mathbb{E}(tr\chi_\Lambda E_{H_{T_i\omega}}(\cdot)\chi_\Lambda) \tag{13}$$

and from (12)

$$\mathbb{E}(tr\chi_\Lambda E_{H_{T_i\omega}}(\cdot)\chi_\Lambda) = \mathbb{E}(tr\chi_\Lambda U_i E_{H_\omega}(\cdot)U_i^*\chi_\Lambda) \tag{14}$$

Let $\{\psi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $L_2(\mathbb{R}^d)$ and denote $\varphi_n := U_i^*\psi_n$.

$$\begin{aligned}
\mathbb{E}(tr\chi_\Lambda U_i E_{H_\omega}(\cdot)U_i^*\chi_\Lambda) &= \mathbb{E}\left(\sum_n \langle\psi_n, \chi_\Lambda U_i E_{H_\omega}(\cdot)U_i^*\chi_\Lambda\psi_n\rangle\right) \\
&= \mathbb{E}\left(\sum_n \langle U_i^*\chi_\Lambda\psi_n, E_{H_\omega}(\cdot)U_i^*\chi_\Lambda\psi_n\rangle\right) \\
&= \mathbb{E}\left(\sum_n \langle \chi_\Lambda\varphi_n, E_{H_\omega}(\cdot)\chi_\Lambda\varphi_n\rangle\right) \\
&= \mathbb{E}\left(\sum_n \langle \chi_{\Lambda_i}\psi_n, E_{H_\omega}(\cdot)\chi_{\Lambda_i}\psi_n\rangle\right) \\
&= \mathbb{E}(tr\chi_{\Lambda_i}E_{H_\omega}(\cdot)\chi_{\Lambda_i})
\end{aligned} \tag{15}$$

Then from (13)-(15) , assertion a) of the Lemma follows.

Case b). Discrete model.

$$\begin{aligned}
\mathbb{E}(\langle\delta_0, E_{H_\omega}(\cdot)\delta_0\rangle) &= \mathbb{E}(\langle\delta_0, E_{H_{T_i\omega}}(\cdot)\delta_0\rangle) \\
&= \mathbb{E}(\langle\delta_0, U_i E_{H_\omega}(\cdot)U_i^*\delta_0\rangle) \\
&= \mathbb{E}(\langle U_i^*\delta_0, E_{H_\omega}(\cdot)U_i^*\delta_0\rangle) \\
&= \mathbb{E}(\langle\delta_{-i}, E_{H_\omega}(\cdot)\delta_{-i}\rangle)
\end{aligned}$$

The first equality follows from (11) and the second from (12). (In fact from [15] and [5] $\mathbb{P}$ almost surely the measures $\langle \delta_j, E_{H_\omega}(\cdot)\delta_j \rangle)$ are equivalent for all j, in many cases.) $\square$

Lemma 2. *Let S be a selfadjoint operator in a Hilbert space $\mathcal{H}$ and M a closed subspace of $\mathcal{H}$ which is a reducing subspace for S. Let $S_M := S|_M$ be restriction of S to M. Then S_M is selfadjoint and for $t \in \mathbb{R}$*

$$E_S(t)|_M = E_{S_M}(t)$$

where $E_S(t)$ and $E_{S_M}(t)$ denote the spectral families of orthogonal projections given by the spectral theorem associated to the operators S and S_M respectively. $E_S(t)|_M$ denotes the restriction of $E_S(t)$ to the subspace M.

Proof. The restriction of $E_S(t)$ to the subspace M, denoted by $E_S\big|_M(t)$, is a spectral family in the Hilbert Space M.

The properties required (see f.e. [24] section 7.2):

i) $E_S|_M(t)^2 = E_S|_M(t)$ and $E_S|_M(t) = E_S|_M(t)^*$, for every $t \in \mathbb{R}$

ii) $E_S|_M(s) \leq E_S|_M(t)$ for $s \leq t$ (monotonicity)

iii)$E_S|_M(t+\varepsilon) \longrightarrow E_S|_M(t)$ for all $t \in \mathbb{R}$ as $\varepsilon \longrightarrow 0+$ (continuity from the right)

iv)$E_S|_M(t)g \longrightarrow 0$ for every $g \in \mathcal{H}$ as $t \longrightarrow -\infty$, $E_S|_M(t)g \longrightarrow Id$ for every $g \in \mathcal{H}$ as $t \longrightarrow \infty$.

follow from the corresponding properties for $E_S(t)$.

According to (see [24] thm. 7.28), the operator S_M is selfadjoint. There exists therefore a corresponding spectral family $E_{S_M}(t)$ such that $S_M f = \int \lambda dE_{S_M}(\lambda)f$ where by definition the vector $\int \lambda dE_{S_M}(\lambda)f$ is the one such that $\langle \int \lambda dE_{S_M}(\lambda)f, v\rangle = \int \lambda d\langle E_{S_M}(\lambda)f, v\rangle$ for $v \in M$ see [8]p. 356. Therefore we have for $f \in D(S) \cap M$

$$\begin{aligned}
\langle \int \lambda dE_{S_M}(\lambda)f, v\rangle &= \langle S_M f, v\rangle = \langle \int \lambda dE_S(\lambda)f, v\rangle \\
&= \int \lambda d\langle E_S(\lambda)f, v\rangle = \int \lambda d\langle E_S|_M f, v\rangle \\
&= \langle \int \lambda d(E_S(\lambda)|_M)f, v\rangle
\end{aligned}$$

Since the spectral family associated to a selfadjoint operator is unique, see [24] thm 7.17, we obtain

$$E_S(t)f = E_S(t)|_M f = E_{S_M}(t)f$$

if $f \in M$.

$\square$

Denote by $RangT = \{Tf \mid f \in D(T)\}$ the range of the operator T.

In the following Lemma we use the same notation as in Lemma 2.

Lemma 3. *It is possible to choose $M = RangE_S(I)$ as a reducing subspace for S in Lemma 2. Here I is a closed interval $I = \{x \in \mathbb{R} \mid a \leq x \leq b\}$, which could be unbounded ($a = -\infty$ or $b = \infty$ are allowed). In this case*

$$\sigma(S) \cap I^\circ \subset \sigma(S_M) \subset \sigma(S) \cap I$$

where I° denotes the interior of I

Proof. $RangE_S(I)$ is a reducing subspace because $E_S(I)E_S(t) = E_S(t)E_S(I)$ for all $t \in \mathbb{R}$ and a space is a reducing subspace if and only if the projection on this space commutes with the spectral family of the operator. See [24] theorem 7.28.

Recall that for a selfadjoint operator T,

$$\lambda \in \sigma(T) \quad \text{if and only if} \quad E_T((\lambda - \varepsilon, \lambda + \varepsilon)) \neq 0$$

. for every $\varepsilon > 0$. See for example [20] thm 3.8. By Lemma 2,

$$E_{S_M}((\lambda - \varepsilon, \lambda + \varepsilon)) = E_S((\lambda - \varepsilon, \lambda + \varepsilon))|_M$$

where $M = RangE_S(I)$. Therefore to find the spectrum $\sigma(S_M)$ of S_M we may look for the points λ for which there exists $g_\varepsilon \in M = RangeE_S(I)$ such that $E_S((\lambda - \varepsilon, \lambda + \varepsilon))g_\varepsilon \neq 0$ for every $\varepsilon > 0$ Therefore

$$\lambda \in \sigma(S_M) \Longleftrightarrow E_S((\lambda - \varepsilon, \lambda + \varepsilon))E_S(I)f_\varepsilon = E_S((\lambda - \varepsilon, \lambda + \varepsilon) \cap I)f_\varepsilon \neq 0$$

for some $f_\varepsilon \in \mathcal{H}$, for every $\varepsilon > 0$, that is

$$\lambda \in \sigma(S_M) \Longleftrightarrow E_S((\lambda - \varepsilon, \lambda + \varepsilon) \cap I) \neq 0$$

for every $\varepsilon > 0$.

Assume $\lambda \in I^\circ \cap \sigma(S)$. Take $\varepsilon > 0$ such that $(\lambda - \varepsilon, \lambda + \varepsilon) \subset I$. Since $\lambda \in \sigma(S)$ then

$$E_S((\lambda - \varepsilon, \lambda + \varepsilon) \cap I) = E_S((\lambda - \varepsilon, \lambda + \varepsilon)) \neq 0$$

It can be seen this happens for every $\varepsilon > 0$ and then we obtain

$$\sigma(S) \cap I^{\circ} \subset \sigma(S_M)$$

.

If $\lambda \notin I$, then there exists $\varepsilon > 0$ such that $(\lambda - \varepsilon, \lambda + \varepsilon) \cap I = \emptyset$. Then $E_S((\lambda - \varepsilon, \lambda + \varepsilon) \cap I) = E_S(\emptyset) = 0$ and $\lambda \notin \sigma(S_M)$. Therefore $\sigma(S_M) \subset I$. Since $E_S((\lambda - \varepsilon, \lambda + \varepsilon) \cap I) \neq 0$ implies $E_S((\lambda - \varepsilon, \lambda + \varepsilon)) \neq 0$, then we have $\sigma(S_M) \subset \sigma(S) \cap I$, and the Lemma is proved.

□

4 Examples

Here we shall consider situations where the IDS is absolutely continuous and therefore Theorem 4 can be applied to obtain that $|\Sigma \cap J| = 0$ implies $\Sigma \cap J = \emptyset$, where Σ is an almost sure spectrum and J a closed interval. We do not intend to be exhaustive and just mention some of the interesting cases.

In [6] random Schrödinger operators of the form $H_\omega(\lambda) = H_0 + \lambda V_\omega$ on $L_2(\mathbb{R}^d)$ are considered for $\lambda \in \mathbb{R}$. The operator $H_0 = (-i\nabla - A_0)^2 + V_0$ is nonrandom. The random Anderson type potential V_ω is constructed from the nonzero single-site potential $u \geq 0$ as

$$V_\omega(x) = \sum_{j \in \mathbb{Z}^d} \omega_j u(x - j)$$

where the $\omega_j, j \epsilon \mathbb{Z}^d$ are random variables.

Consider the hypotheses :

H1) The background operator $H_0 = (-i\nabla - A_0)^2 + V_0$ is lower semi-bounded, $\mathbb{Z}^d$-periodic Schrödinger operator with real valued, $\mathbb{Z}^d$-periodic potential V_0 and a $\mathbb{Z}^d$-periodic vector potential A_0. It is assumed that V_0 and A_0 are sufficiently regular so that H_0 is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$.

H2) The periodic operator H_0 has the unique continuation property, that is, for any $E \in \mathbb{R}$ and for any function $\phi \in H^2_{loc}$, if $(H_0 - E)\phi = 0$ and if ϕ vanishes on an open set, then $\phi \equiv 0$

H3) The nonzero, nonnegative, compactly-supported single-site potential $u \in L_0^\infty(\mathbb{R}^d)$, with $\| u \|_\infty \leq 1$, and it is strictly positive on a nonempty open set.

H4 the random coupling constants $\omega_j, j \epsilon \mathbb{Z}^d$ are independent and identically distributed. The common distribution has density $h_0 \in L_\infty(\mathbb{R})$ with supp $h_0 \subset [0, 1]$.

Then according to [6] Thm. 4.4 and [4] Corollary 1.1). one has the following

Theorem 5. *Assume hypotheses H1)-H4). Then the IDS for the random family $H_\omega(\lambda)$, for $\lambda \neq 0$ is locally Lipschitz continuous on $\mathbb{R}$.*

Therefore the IDS is absolutely continuous and we can apply Theorem 4 as mentioned above. In [6] it is shown that under some other hypothesis there is band-gap localization. This means that the spectrum close to the gaps is pure point almost surely. If we are not near the gaps however, then it is not clear which kind of spectra we have and the fact that the IDS is absolutely continuous guarantees that there is not almost sure spectra of zero measure in this region.

The results just mentioned are quite restrictive about the single site potential u, since it is required to be nonnegative and of compact support. There are results about the absolute continuity of the IDS where these conditions are relaxed. In [22] Thm. 1. for example, single site potentials of *generalized step function form* which are allowed to change sign are considered and in [3] Corollary 4.6. results about the absolute continuity of the IDS are provided, where it is not required the single site potential to be of compact support. In [7] the authors prove the absolute continuity of the integrated density of states for multi-dimensional Schrödinger operators with constant magnetic field and ergodic random potential. Examples of potentials to which these results apply are certain alloy type and Gaussian random potentials. For Gaussian potentials see [23] too.

For the almost periodic case there are results on the absolute continuity of the IDS too. The almost Mathieu operator is defined on $l_2(\mathbb{Z})$ by

$$(Hu)_n = u_{n+1} + u_{n-1} + 2\lambda cos(2\pi[\theta + n\alpha])u_n$$

In [1] it is proven that the integrated density of states of H is absolutely continuous if and only if $|\lambda| \neq 1$. If $|\lambda| < 1$, then the spectral measures of H are absolutely continuous for almost every θ according to [1]. It is known the spectral measures have no absolutely continuous component for $|\lambda| \geq 1$. From our results follow in particular that there is not almost sure singular continuous or pure point spectrum of measure zero for these λ.

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